

# Triangular Hopf Algebras With The Chevalley Property

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## 1 Introduction

Triangular Hopf algebras were introduced by Drinfeld [Dr]. They are the Hopf algebras whose representations form a symmetric tensor category. In that sense, they are the class of Hopf algebras closest to group algebras. The structure of triangular Hopf algebras is far from trivial, and yet is more tractable than that of general Hopf algebras, due to their proximity to groups and Lie algebras. This makes triangular Hopf algebras an excellent testing ground for general Hopf algebraic ideas, methods and conjectures.

A general classification of triangular Hopf algebras is not known yet. However, there are two classes that are relatively well understood. One of them is semisimple triangular Hopf algebras over  $\mathbf{C}$ , for which a complete classification is given in [EG1,EG2]. The key theorem about such Hopf algebras states that each of them is obtained by twisting a group algebra of a finite group (see [EG1, Theorem 2.1]). The proof of this theorem is based on Deligne's theorem on Tannakian categories [De2].

Another important class of Hopf algebras is that of *pointed* ones. These are Hopf algebras whose all simple comodules are 1-dimensional. Theorem 5.1 in [G] gives a classification of minimal triangular pointed Hopf algebras (we note that the additional assumption made in [G, Theorem 5.1] is superfluous, by Theorem 6.1 below).

Recall that a finite-dimensional algebra is called *basic* if all of its simple modules are 1-dimensional (i.e. if its dual is a pointed coalgebra). The same Theorem 5.1 of [G] gives a

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classification of minimal triangular basic Hopf algebras, since the dual of a minimal triangular Hopf algebra is again minimal triangular.

Basic and semisimple Hopf algebras share a common property. Namely, the Jacobson radical  $\text{Rad}(H)$  of such a Hopf algebra  $H$  is a Hopf ideal, and hence the quotient  $H/\text{Rad}(H)$  (the semisimple part) is itself a Hopf algebra. The representation-theoretic formulation of this property is: The tensor product of two simple  $H$ -modules is semisimple. A remarkable classical theorem of Chevalley [C, p.88] states that, over  $\mathbf{C}$ , this property holds for the group algebra of any (not necessarily finite) group. So let us call this property of  $H$  **the Chevalley property**.

The Chevalley property certainly fails for many finite-dimensional Hopf algebras, e.g. for Lusztig's finite-dimensional quantum groups  $U_q(\mathfrak{g})'$  at roots of unity (also known as Frobenius-Lusztig kernels) [L]. However, we found that this property holds for all examples we know of finite-dimensional *triangular* Hopf algebras in characteristic 0. We felt, therefore, that it is natural to classify all finite-dimensional triangular Hopf algebras with the Chevalley property. This is what we do in this paper.

We start by classifying triangular Hopf algebras with  $R$ -matrix of rank  $\leq 2$ . We show that such a Hopf algebra is a suitable modification of a cocommutative Hopf superalgebra (i.e. the group algebra of a supergroup). On the other hand, by a theorem of Kostant [Ko], a finite supergroup is a semidirect product of a finite group with an odd vector space on which this group acts.

Next we prove our main result: Any finite-dimensional triangular Hopf algebra with the Chevalley property is obtained by twisting a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ . We also prove the converse result that any such Hopf algebra does have the Chevalley property. As a corollary, we prove that any finite-dimensional triangular Hopf algebra whose coradical is a Hopf subalgebra (e.g. pointed) is obtained by twisting a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ .

The structure of the paper is as follows.

In Section 2 we give the definitions of Hopf superalgebras and twists for them. We also discuss cocommutative Hopf superalgebras, and describe their classification (Kostant's theorem [Ko]).

In Section 3 we establish a correspondence between usual Hopf algebras and Hopf superalgebras, and show how this correspondence extends to twists and to triangular Hopf algebras.

In Section 4 we discuss the Chevalley property.

In Section 5 we prove our main result, and discuss its consequences and some open questions.

In Section 6, using the main theorem, we show that a finite-dimensional cotriangular pointed Hopf algebra is generated by its grouplike and skewprimitive elements. Thus we

confirm the conjecture that this is the case for any finite-dimensional pointed Hopf algebra over  $\mathbf{C}$  [AS2], in the cotriangular case. This allows us to strengthen the main result of [G].

In Section 7 we prove that the categorical dimensions of objects in any abelian symmetric rigid category with finitely many irreducible objects are integers. In particular, this is the case for the representation category of a triangular Hopf algebra. This gives supporting evidence for a positive answer to the question we ask in Section 5: Is any finite-dimensional triangular Hopf algebra a twist of a modified supergroup algebra?

In the appendix we use the Lifting method [AS1,AS2] to give other proofs of Theorem 5.2.1 and Corollary 6.3, and a generalization of Lemma 5.3.4.

We note that similarly to the case of semisimple Hopf algebras, the proof of our main result is based on Deligne's theorem [De2]. In fact, we use Theorem 2.1 of [EG1] to prove the main result of this paper.

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## 2 Hopf Superalgebras

### 2.1 Supervector Spaces

The ground field in this paper will always be the field  $\mathbf{C}$  of complex numbers.

We start by recalling the definition of the category of supervector spaces. A Hopf algebraic way to define this category is as follows.

Let  $u$  be the generator of the group  $\mathbb{Z}_2$  of two elements, and set

$$R_u := \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u) \in \mathbf{C}[\mathbb{Z}_2] \otimes \mathbf{C}[\mathbb{Z}_2]. \quad (1)$$

Then  $(\mathbf{C}[\mathbb{Z}_2], R_u)$  is a minimal triangular Hopf algebra.

**Definition 2.1.1** *The category of supervector spaces over  $\mathbf{C}$  is the symmetric tensor category  $\text{Rep}(\mathbf{C}[\mathbb{Z}_2], R_u)$  of representations of the triangular Hopf algebra  $(\mathbf{C}[\mathbb{Z}_2], R_u)$ . This category will be denoted by  $\text{SuperVect}$ .*

For  $V \in \text{SuperVect}$  and  $v \in V$ , we say that  $v$  is even if  $uv = v$  and odd if  $uv = -v$ . The set of even vectors in  $V$  is denoted by  $V_0$  and the set of odd vectors by  $V_1$ , so  $V = V_0 \oplus V_1$ . We define the parity of a vector  $v$  to be  $p(v) = 0$  if  $v$  is even and  $p(v) = 1$  if  $v$  is odd (if  $v$  is neither odd nor even,  $p(v)$  is not defined).

Thus, as an ordinary tensor category, SuperVect is equivalent to the category of representations of  $\mathbb{Z}_2$ , but the commutativity constraint is different from that of  $\text{Rep}(\mathbb{Z}_2)$  and equals  $\beta := R_u P$ , where  $P$  is the permutation of components. In other words, we have

$$\beta(v \otimes w) = (-1)^{p(v)p(w)} w \otimes v, \quad (2)$$

where both  $v, w$  are either even or odd.

## 2.2 Hopf Superalgebras

Recall that in any symmetric (more generally, braided) tensor category, one can define an algebra, coalgebra, bialgebra, Hopf algebra, triangular Hopf algebra, etc, to be an object of this category equipped with the usual structure maps (morphisms in this category), subject to the same axioms as in the usual case. In particular, any of these algebraic structures in the category SuperVect is usually identified by the prefix “super”. For example:

**Definition 2.2.1** *A Hopf superalgebra is a Hopf algebra in SuperVect.*

More specifically, a Hopf superalgebra  $\mathcal{H}$  is an ordinary  $\mathbb{Z}_2$ -graded associative unital algebra with multiplication  $m$ , equipped with a coassociative map  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  (a morphism in SuperVect) which is multiplicative in the super-sense, and with a counit and antipode satisfying the standard axioms. Here multiplicativity in the super-sense means that  $\Delta$  satisfies the relation

$$\Delta(ab) = \sum (-1)^{p(a_2)p(b_1)} a_1 b_1 \otimes a_2 b_2 \quad (3)$$

for all  $a, b \in \mathcal{H}$  (where  $\Delta(a) = \sum a_1 \otimes a_2$ ,  $\Delta(b) = \sum b_1 \otimes b_2$ ). This is because the tensor product of two algebras  $A, B$  in SuperVect is defined to be  $A \otimes B$  as a vector space, with multiplication

$$(a \otimes b)(a' \otimes b') := (-1)^{p(a')p(b)} aa' \otimes bb'. \quad (4)$$

**Remark 2.2.2** Hopf superalgebras appear in [Ko], under the name of “graded Hopf algebras”.

Similarly, a (quasi)triangular Hopf superalgebra  $(\mathcal{H}, \mathcal{R})$  is a Hopf superalgebra with an  $R$ -matrix (an even element  $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ ) satisfying the usual axioms. As in the even case, an important role is played by the Drinfeld element  $u$  of  $(\mathcal{H}, \mathcal{R})$ :

$$u := m \circ \beta \circ (Id \otimes S)(\mathcal{R}). \quad (5)$$

For instance,  $(\mathcal{H}, \mathcal{R})$  is triangular if and only if  $u$  is a grouplike element of  $\mathcal{H}$ .

As in the even case, the tensorands of the  $R$ -matrix of a (quasi)triangular Hopf superalgebra  $\mathcal{H}$  generate a finite-dimensional sub Hopf superalgebra  $\mathcal{H}_m$ , called the *minimal part of  $\mathcal{H}$*  (the proof does not differ essentially from the proof of the analogous fact for Hopf algebras). A (quasi)triangular Hopf superalgebra is said to be minimal if it coincides with its minimal part. The dimension of the minimal part is the *rank* of the  $R$ -matrix.

## 2.3 Cocommutative Hopf Superalgebras

**Definition 2.3.1** *We will say that a Hopf superalgebra  $\mathcal{H}$  is commutative (resp. cocommutative) if  $m = m \circ \beta$  (resp.  $\Delta = \beta \circ \Delta$ ).*

**Example 2.3.2 [Ko]** Let  $G$  be a group, and  $\mathfrak{g}$  a Lie superalgebra with an action of  $G$  by automorphisms of Lie superalgebras. Let  $\mathcal{H} := \mathbf{C}[G] \ltimes U(\mathfrak{g})$ , where  $U(\mathfrak{g})$  denotes the universal enveloping algebra of  $\mathfrak{g}$ . Then  $\mathcal{H}$  is a cocommutative Hopf superalgebra, with  $\Delta(x) = x \otimes 1 + 1 \otimes x$ ,  $x \in \mathfrak{g}$ , and  $\Delta(g) = g \otimes g$ ,  $g \in G$ . In this Hopf superalgebra, we have  $S(g) = g^{-1}$ ,  $S(x) = -x$ , and in particular  $S^2 = Id$ .

The Hopf superalgebra  $\mathcal{H}$  is finite-dimensional if and only if  $G$  is finite, and  $\mathfrak{g}$  is finite-dimensional and purely odd (and hence commutative). Then  $\mathcal{H} = \mathbf{C}[G] \ltimes \Lambda V$ , where  $V = \mathfrak{g}$  is an odd vector space with a  $G$ -action. In this case,  $\mathcal{H}^*$  is a commutative Hopf superalgebra.

**Remark 2.3.3** We note that as in the even case, it is convenient to think about  $\mathcal{H}$  and  $\mathcal{H}^*$  in geometric terms. Consider, for instance, the finite-dimensional case. In this case, it is useful to think of the “affine algebraic supergroup”  $\tilde{G} := G \ltimes V$ . Then one can regard  $\mathcal{H}$  as the group algebra  $\mathbf{C}[\tilde{G}]$  of this supergroup, and  $\mathcal{H}^*$  as its function algebra  $F(\tilde{G})$ . Having this in mind, we will call the algebra  $\mathcal{H}$  a **supergroup algebra**.

It turns out that like in the even case, any cocommutative Hopf superalgebra is of the type described in Example 2.3.2. Namely, we have the following theorem.

**Theorem 2.3.4 ([Ko], Theorem 3.3)** *Let  $\mathcal{H}$  be a cocommutative Hopf superalgebra over  $\mathbf{C}$ . Then  $\mathcal{H} = \mathbf{C}[\mathbf{G}(\mathcal{H})] \ltimes U(\mathbf{P}(\mathcal{H}))$ , where  $U(\mathbf{P}(\mathcal{H}))$  is the universal enveloping algebra of the Lie superalgebra of primitive elements of  $\mathcal{H}$ , and  $\mathbf{G}(\mathcal{H})$  is the group of grouplike elements of  $\mathcal{H}$ .*

In particular, in the finite-dimensional case we get:

**Corollary 2.3.5** *Let  $\mathcal{H}$  be a finite-dimensional cocommutative Hopf superalgebra over  $\mathbf{C}$ . Then  $\mathcal{H} = \mathbf{C}[\mathbf{G}(\mathcal{H})] \ltimes \Lambda V$ , where  $V$  is the space of primitive elements of  $\mathcal{H}$  (regarded as an odd vector space) and  $\mathbf{G}(\mathcal{H})$  is the finite group of grouplikes of  $\mathcal{H}$ . In other words,  $\mathcal{H}$  is a supergroup algebra. ■*

This corollary will be used below, and although it follows at once from Theorem 2.3.4, for the sake of completeness we will give its proof in Section 5.

## 2.4 Twists

A twist for a Hopf algebra in any symmetric tensor category is defined in the same way as in the usual case (see [Dr]). However, for the reader's convenience, we will repeat this definition (for Hopf superalgebras).

Let  $\mathcal{H}$  be a Hopf superalgebra. The multiplication, unit, comultiplication, counit and antipode in  $\mathcal{H}$  will be denoted by  $m, 1, \Delta, \varepsilon, S$  respectively.

**Definition 2.4.1** *A twist for  $\mathcal{H}$  is an invertible even element  $\mathcal{J} \in \mathcal{H} \otimes \mathcal{H}$  which satisfies*

$$(\Delta \otimes Id)(\mathcal{J})(\mathcal{J} \otimes 1) = (Id \otimes \Delta)(\mathcal{J})(1 \otimes \mathcal{J}) \text{ and } (\varepsilon \otimes Id)(\mathcal{J}) = (Id \otimes \varepsilon)(\mathcal{J}) = 1, \quad (6)$$

where  $Id$  is the identity map of  $\mathcal{H}$ .

Given a twist  $\mathcal{J}$  for  $\mathcal{H}$ , one can define a new Hopf superalgebra structure

$$(\mathcal{H}^{\mathcal{J}}, m, 1, \Delta^{\mathcal{J}}, \varepsilon, S^{\mathcal{J}})$$

on the algebra  $(\mathcal{H}, m, 1)$  as follows. The coproduct is determined by

$$\Delta^{\mathcal{J}}(a) = \mathcal{J}^{-1} \Delta(a) \mathcal{J} \text{ for any } a \in \mathcal{H}, \quad (7)$$

and the antipode is determined by

$$S^{\mathcal{J}}(a) = Q^{-1} S(a) Q \text{ for any } a \in \mathcal{H}, \quad (8)$$

where  $Q := m \circ (S \otimes Id)(\mathcal{J})$ .

If  $\mathcal{H}$  is (quasi)triangular with the universal  $R$ -matrix  $\mathcal{R}$  then so is  $\mathcal{H}^{\mathcal{J}}$  with the universal  $R$ -matrix  $\mathcal{R}^{\mathcal{J}} := \mathcal{J}_{21}^{-1} \mathcal{R} \mathcal{J}$ .

## 3 Triangular Hopf Algebras With Drinfeld Element of Order $\leq 2$

### 3.1 The Correspondence Between Hopf Algebras and Superalgebras

We can now prove our first results, which will be essential in the next section. We start with a correspondence theorem between Hopf algebras and Hopf superalgebras.

**Theorem 3.1.1** *There is a one to one correspondence between:*

1. isomorphism classes of pairs  $(H, u)$  where  $H$  is an ordinary Hopf algebra, and  $u$  is a grouplike element in  $H$  such that  $u^2 = 1$ , and
2. isomorphism classes of pairs  $(\mathcal{H}, g)$  where  $\mathcal{H}$  is a Hopf superalgebra, and  $g$  is a grouplike element in  $\mathcal{H}$  such that  $g^2 = 1$  and  $gxg^{-1} = (-1)^{p(x)}x$  (i.e.  $g$  acts on  $x$  by its parity),

such that the tensor categories of representations of  $H$  and  $\mathcal{H}$  are equivalent.

**Proof:** Let  $(H, u)$  be an ordinary Hopf algebra with comultiplication  $\Delta$ , counit  $\varepsilon$ , antipode  $S$ , and a grouplike element  $u$  such that  $u^2 = 1$ . Let  $\mathcal{H} = H$  regarded as a superalgebra, where the  $\mathbb{Z}_2$ -grading is given by the adjoint action of  $u$ . For  $h \in H$ , let us define  $\Delta_0, \Delta_1$  by writing  $\Delta(h) = \Delta_0(h) + \Delta_1(h)$ , where  $\Delta_0(h) \in H \otimes H_0$  and  $\Delta_1(h) \in H \otimes H_1$ . Define a map  $\tilde{\Delta} : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  by  $\tilde{\Delta}(h) := \Delta_0(h) - (-1)^{p(h)}(u \otimes 1)\Delta_1(h)$ . Define  $\tilde{S}(h) := u^{p(h)}S(h)$ ,  $h \in H$ . Then it is straightforward to verify that  $(\mathcal{H}, \tilde{\Delta}, \varepsilon, \tilde{S})$  is a Hopf superalgebra.

The element  $u$  remains grouplike in the new Hopf superalgebra, and acts by parity, so we can set  $g := u$ .

Conversely, suppose that  $(\mathcal{H}, g)$  is a pair where  $\mathcal{H}$  is a Hopf superalgebra with comultiplication  $\tilde{\Delta}$ , counit  $\varepsilon$ , antipode  $\tilde{S}$ , and a grouplike element  $g$ , with  $g^2 = 1$ , acting by parity. For  $h \in \mathcal{H}$ , let us define  $\tilde{\Delta}_0, \tilde{\Delta}_1$  by writing  $\tilde{\Delta}(h) = \tilde{\Delta}_0(h) + \tilde{\Delta}_1(h)$ , where  $\tilde{\Delta}_0(h) \in \mathcal{H} \otimes \mathcal{H}_0$  and  $\tilde{\Delta}_1(h) \in \mathcal{H} \otimes \mathcal{H}_1$ . Let  $H = \mathcal{H}$  as algebras, and define a map  $\Delta : H \rightarrow H \otimes H$  by  $\Delta(h) := \tilde{\Delta}_0(h) - (-1)^{p(h)}(g \otimes 1)\tilde{\Delta}_1(h)$ . Define  $S(h) := g^{p(h)}\tilde{S}(h)$ ,  $h \in H$ . Then it is straightforward to verify that  $(H, \Delta, \varepsilon, S)$  is an ordinary Hopf algebra, and we can set  $u := g$ .

It is obvious that the two assignments constructed above are inverse to each other. The equivalence of tensor categories is straightforward to verify. The theorem is proved. ■

Theorem 3.1.1 implies the following. Let  $\mathcal{H}$  be *any* Hopf superalgebra, and  $\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}$  be the semidirect product, where the generator  $g$  of  $\mathbb{Z}_2$  acts on  $\mathcal{H}$  by  $gxg^{-1} = (-1)^{p(x)}x$ . Then we can define an ordinary Hopf algebra  $\overline{\mathcal{H}}$ , which is the one corresponding to  $(\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}, g)$  under the correspondence of Theorem 3.1.1.

The constructions of this section have the following explanation in terms of Radford's biproduct construction [R2]. Namely  $\mathcal{H}$  is a Hopf algebra in the Yetter-Drinfeld category of  $\mathbf{C}[\mathbb{Z}_2]$ , so Radford's biproduct construction yields a Hopf algebra structure on  $\mathbf{C}[\mathbb{Z}_2] \otimes \mathcal{H}$ , and it is straightforward to see that this Hopf algebra is exactly  $\overline{\mathcal{H}}$ . Moreover, it is clear that for any pair  $(H, u)$  as in Theorem 3.1.1,  $gu$  is central in  $\overline{\mathcal{H}}$  and  $H = \overline{\mathcal{H}}/(gu - 1)$ .

Let us give an interesting corollary of Theorem 3.1.1, even though we will not use it.

**Corollary 3.1.2** *Let  $\mathcal{H}$  be a finite-dimensional Hopf superalgebra over  $\mathbf{C}$ . Then:*

1.  $\mathcal{H}$  is semisimple if and only if it is cosemisimple.
2. If  $\mathcal{H}$  is semisimple then  $S^4 = \text{Id}$ .

3. If  $\mathcal{H}$  is semisimple and  $S^2 = Id$ , then  $\mathcal{H}$  is purely even, i.e. it is a usual semisimple Hopf algebra.

**Proof:** 1. If  $\mathcal{H}$  is semisimple then so is  $\overline{\mathcal{H}}$ , hence so is  $(\overline{\mathcal{H}})^*$ . But it is easy to show that  $(\overline{\mathcal{H}})^*$  is isomorphic as an algebra to  $\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}^*$  (unlike the dual of  $\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}$ , which is isomorphic to  $\mathbf{C}[\mathbb{Z}_2] \otimes \mathcal{H}^*$ ). Thus this crossed product algebra is semisimple. It is well known (and easy to show) that this implies the semisimplicity of  $\mathcal{H}^*$ .

2. The Hopf algebra  $\overline{\mathcal{H}}$  is semisimple, so we have  $S^2 = Id$  in it. Thus, in  $\mathcal{H}$  we have  $S^2 = \text{Ad}(g)$ , so  $S^4 = \text{Ad}(g^2) = Id$ .

3. Since  $S^2 = \text{Ad}(g)$ ,  $g$  has to be central. Thus,  $\mathcal{H}$  is purely even. ■

**Remark 3.1.3** The example of supergroup algebras shows that for finite-dimensional Hopf superalgebras, unlike usual Hopf algebras,  $S^2 = Id$  does not imply semisimplicity or cosemisimplicity. In fact, Corollary 3.1.2, part 3, shows that, in a sense, the situation is exactly the opposite.

## 3.2 Correspondence of Twists

Let us say that a twist  $J$  for a Hopf algebra  $H$  with an involutive grouplike element  $g$  is *even* if it is invariant under  $\text{Ad}(g)$ .

**Proposition 3.2.1** *Let  $(\mathcal{H}, g)$  be a pair as in Theorem 3.1.1, and let  $H$  be the associated ordinary Hopf algebra. Let  $\mathcal{J} \in \mathcal{H} \otimes \mathcal{H}$  be an even element. Write  $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$ , where  $\mathcal{J}_0 \in \mathcal{H}_0 \otimes \mathcal{H}_0$  and  $\mathcal{J}_1 \in \mathcal{H}_1 \otimes \mathcal{H}_1$ . Define  $J := \mathcal{J}_0 - (g \otimes 1)\mathcal{J}_1$ . Then  $J$  is an even twist for  $H$  if and only if  $\mathcal{J}$  is a twist for  $\mathcal{H}$ . Moreover,  $\mathcal{H}^{\mathcal{J}}$  corresponds to  $H^J$  under the correspondence in Theorem 3.1.1. Thus, there is a one to one correspondence between even twists for  $H$  and twists for  $\mathcal{H}$ , given by  $J \rightarrow \mathcal{J}$ .*

**Proof:** Straightforward. ■

## 3.3 The Correspondence Between Triangular Hopf Algebras and Superalgebras

Let us now return to our main subject, which is triangular Hopf algebras and superalgebras. For triangular Hopf algebras whose Drinfeld element  $u$  is involutive, we will make the natural choice of the element  $u$  in Theorem 3.1.1, namely define it to be the Drinfeld element of  $H$ .

**Theorem 3.3.1** *The correspondence of Theorem 3.1.1 extends to a one to one correspondence between:*

1. isomorphism classes of ordinary triangular Hopf algebras  $H$  with Drinfeld element  $u$  such that  $u^2 = 1$ , and
2. isomorphism classes of pairs  $(\mathcal{H}, g)$  where  $\mathcal{H}$  is a triangular Hopf superalgebra with Drinfeld element 1 and  $g$  is an element of  $\mathbf{G}(\mathcal{H})$  such that  $g^2 = 1$  and  $gxg^{-1} = (-1)^{p(x)}x$ .

**Proof:** Let  $(H, R)$  be a triangular Hopf algebra with  $u^2 = 1$ . Since  $(S \otimes S)(R) = R$  and  $S^2 = \text{Ad}(u)$  [Dr],  $u \otimes u$  and  $R$  commute. Hence we can write  $R = R_0 + R_1$ , where  $R_0 \in H_0 \otimes H_0$  and  $R_1 \in H_1 \otimes H_1$ . Let  $\mathcal{R} := (R_0 + (1 \otimes u)R_1)R_u$ . Then  $\mathcal{R}$  is even. Indeed, since  $R_0 = 1/2(R + (u \otimes 1)R(u \otimes 1))$  and  $R_1 = 1/2(R - (u \otimes 1)R(u \otimes 1))$ ,  $u \otimes u$  and  $\mathcal{R}$  commute.

It is now straightforward to show that  $(\mathcal{H}, \mathcal{R})$  is triangular with Drinfeld element 1. Let us show for instance that  $\mathcal{R}$  is unitary. Let us use the notation  $a * b, X^{21}$  for multiplication and opposition in the tensor square of a superalgebra, and the notation  $ab, X^{op}$  for usual algebras. Then,

$$\mathcal{R} * \mathcal{R}_{21} = (R_0 + (1 \otimes u)R_1)R_u * (R_0^{op} - (u \otimes 1)R_1^{op})R_u.$$

Since,  $R_u R_0 = R_0 R_u$ ,  $R_u R_1 = -(u \otimes u)R_1 R_u$ , we get that the RHS equals

$$(R_0 + (1 \otimes u)R_1) * (R_0^{op} + (1 \otimes u)R_1^{op}) = R_0 R_0^{op} + R_1 R_1^{op} + (1 \otimes u)(R_1 R_0^{op} + R_0 R_1^{op}).$$

But,  $R_0 R_0^{op} + R_1 R_1^{op} = 1$  and  $(1 \otimes u)(R_1 R_0^{op} + R_0 R_1^{op}) = 0$ , since  $RR^{op} = 1$ , so we are done.

Conversely, suppose that  $(\mathcal{H}, g)$  is a pair where  $\mathcal{H}$  is a triangular Hopf superalgebra with  $R$ -matrix  $\mathcal{R}$  and Drinfeld element 1. Let  $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$ , where  $\mathcal{R}_0$  has even components, and  $\mathcal{R}_1$  has odd components. Let  $R := (\mathcal{R}_0 + (1 \otimes g)\mathcal{R}_1)R_g$ . Then it is straightforward to show that  $(H, R)$  is triangular with Drinfeld element  $u = g$ . The theorem is proved. ■

**Corollary 3.3.2** *If  $(\mathcal{H}, \mathcal{R})$  is a triangular Hopf superalgebra with Drinfeld element 1, then the Hopf algebra  $\overline{\mathcal{H}}$  is also triangular, with the  $R$ -matrix*

$$\overline{R} := (\mathcal{R}_0 + (1 \otimes g)\mathcal{R}_1)R_g, \tag{9}$$

*where  $g$  is the grouplike element adjoined to  $\mathcal{H}$  to obtain  $\overline{\mathcal{H}}$ . Moreover,  $\mathcal{H}$  is minimal if and only if so is  $\overline{\mathcal{H}}$ .*

**Proof:** Clear. ■

The following corollary, combined with Kostant's theorem, gives a classification of triangular Hopf algebras with  $R$ -matrix of rank  $\leq 2$  (i.e. of the form  $R_u$  as in (1), where  $u$  is a grouplike of order  $\leq 2$ ).

**Corollary 3.3.3** *The correspondence of Theorem 3.3.1 restricts to a one to one correspondence between:*

1. *isomorphism classes of ordinary triangular Hopf algebras with  $R$ -matrix of rank  $\leq 2$ , and*
2. *isomorphism classes of pairs  $(\mathcal{H}, g)$  where  $\mathcal{H}$  is a cocommutative Hopf superalgebra and  $g$  is an element of  $\mathbf{G}(\mathcal{H})$  such that  $g^2 = 1$  and  $gxg^{-1} = (-1)^{p(x)}x$ .*

**Proof:** Let  $(H, R)$  be an ordinary triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ . In particular, the Drinfeld element  $u$  of  $H$  satisfies  $u^2 = 1$ , and  $R = R_u$ . Hence by Theorem 3.3.1,  $(\mathcal{H}, \tilde{\Delta}, \mathcal{R})$  is a triangular Hopf superalgebra. Moreover, it is cocommutative since  $\mathcal{R} = R_u R_u = 1$ .

Conversely, for any  $(\mathcal{H}, g)$ , by Theorem 3.3.1, the pair  $(H, R_g)$  is an ordinary triangular Hopf algebra, and clearly the rank of  $R_g$  is  $\leq 2$ . ■

In particular, Corollaries 2.3.5 and 3.3.3 imply that finite-dimensional triangular Hopf algebras with  $R$ -matrix of rank  $\leq 2$  correspond to supergroup algebras. In view of this, we make the following definition.

**Definition 3.3.4** *A finite-dimensional triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$  is called a modified supergroup algebra.*

### 3.4 Construction of Twists for Supergroup Algebras

**Proposition 3.4.1** *Let  $\mathcal{H} = \mathbf{C}[G] \ltimes \Lambda V$  be a supergroup algebra. Let  $r \in S^2 V$ . Then  $\mathcal{J} := e^{r/2}$  is a twist for  $\mathcal{H}$ . Moreover,  $((\Lambda V)^{\mathcal{J}}, \mathcal{J}_{21}^{-1} \mathcal{J})$  is minimal triangular if and only if  $r$  is nondegenerate.*

**Proof:** Straightforward. ■

**Example 3.4.2** Let  $G$  be the group of order 2 with generator  $g$ . Let  $V := \mathbf{C}$  be the nontrivial 1-dimensional representation of  $G$ , and write  $\Lambda V = sp\{1, x\}$ . Then the associated ordinary triangular Hopf algebra to  $(\mathcal{H}, g) := (\mathbf{C}[G] \ltimes \Lambda V, g)$  is Sweedler's 4-dimensional Hopf algebra  $H$  [S] with the triangular structure  $R_g$ . Namely, the algebra  $H$  is generated by a grouplike element  $g$  and a  $1 : g$  skew primitive element  $x$  (i.e.  $\Delta(x) = x \otimes 1 + g \otimes x$ ) satisfying the relations  $g^2 = 1$ ,  $x^2 = 0$  and  $gx = -xg$ . It is known [R1] that the set of triangular structures on  $H$  is parameterized by  $\mathbf{C}$ ; namely,  $R$  is a triangular structure on  $H$  if and only if

$$R = R_\lambda := R_g - \frac{\lambda}{2}(x \otimes x - gx \otimes x + x \otimes gx + gx \otimes gx), \quad \lambda \in \mathbf{C}.$$

Clearly,  $(H, R_\lambda)$  is minimal if and only if  $\lambda \neq 0$ .

Let  $r \in S^2V$  be defined by  $r := \lambda x \otimes x$ ,  $\lambda \in \mathbf{C}$ . Set  $\mathcal{J}_\lambda := e^{r/2} = 1 + \frac{1}{2}\lambda x \otimes x$ ; it is a twist for  $\mathcal{H}$ . Hence,  $J_\lambda := 1 - \frac{1}{2}\lambda g x \otimes x$  is a twist for  $H$ . It is easy to check that  $R_\lambda = (J_\lambda)_{21}^{-1} R_g J_\lambda$ . Thus,  $(H, R_\lambda) = (H, R_0)^{J_\lambda}$ .

**Remark 3.4.3** In fact, Radford's classification of triangular structures on  $H$  can be easily deduced from Lemma 5.3.4 below.

## 4 The Chevalley Property

Recall that in the introduction we made the following definition.

**Definition 4.1** *A Hopf algebra  $H$  over  $\mathbf{C}$  is said to have the Chevalley property if the tensor product of any two simple  $H$ -modules is semisimple. More generally, let us say that a tensor category has the Chevalley property if the tensor product of two simple objects is semisimple.*

Let us give some equivalent formulations of the Chevalley property.

**Proposition 4.2** *Let  $H$  be a finite-dimensional Hopf algebra over  $\mathbf{C}$  and let  $A := H^*$ . The following conditions are equivalent:*

1.  $H$  has the Chevalley property.
2. The category of (right)  $A$ -comodules has the Chevalley property.
3.  $\text{Corad}(A)$  is a Hopf subalgebra of  $A$ .
4.  $\text{Rad}(H)$  is a Hopf ideal and thus  $H/\text{Rad}(H)$  is a Hopf algebra.
5.  $S^2 = \text{Id}$  on  $H/\text{Rad}(H)$ , or equivalently on  $\text{Corad}(A)$ .

**Proof:** (1.  $\Leftrightarrow$  2.) Clear, since the categories of left  $H$ -modules and right  $A$ -comodules are equivalent.

(2.  $\Rightarrow$  3.) Recall the definition of a matrix coefficient of a comodule  $V$  over  $A$ . If  $\rho : V \rightarrow V \otimes A$  is the coaction,  $v \in V$ ,  $\alpha \in V^*$ , then

$$\phi_{v,\alpha}^V := (\alpha \otimes \text{Id})\rho(v) \in A.$$

It is well-known that:

(a) The coradical of  $A$  is the linear span of the matrix coefficients of all simple  $A$ -comodules.

(b) The product in  $A$  of two matrix coefficients is a matrix coefficient of the tensor product. Specifically,

$$\phi_{v,\alpha}^V \phi_{w,\beta}^W = \phi_{v \otimes w, \alpha \otimes \beta}^{V \otimes W}.$$

It follows at once from (a) and (b) that  $\text{Corad}(A)$  is a subalgebra of  $A$ . Since the coradical is stable under the antipode, the claim follows.

(3.  $\Leftrightarrow$  4.) To say that  $\text{Rad}(H)$  is a Hopf ideal is equivalent to saying that  $\text{Corad}(H^*)$  is a Hopf algebra, since  $\text{Corad}(H^*) = (H/\text{Rad}H)^*$ .

(4.  $\Rightarrow$  1.) If  $V, W$  are simple  $H$ -modules then they factor through  $H/\text{Rad}(H)$ . But  $H/\text{Rad}(H)$  is a Hopf algebra, so  $V \otimes W$  also factors through  $H/\text{Rad}(H)$ , so it is semisimple.

(3.  $\Rightarrow$  5.) Clear, since a cosemisimple Hopf algebra is involutory.

(5.  $\Rightarrow$  3.) Consider the subalgebra  $B$  of  $A$  generated by  $\text{Corad}(A)$ . This is a Hopf algebra, and  $S^2 = \text{Id}$  on it. Thus,  $B$  is cosemisimple and hence  $B = \text{Corad}(A)$  is a Hopf subalgebra of  $A$ . ■

**Remark 4.3** The assumption that the base field has characteristic 0 is needed only in the proof of (5.  $\Leftrightarrow$  3.)

## 5 The Classification of Triangular Hopf Algebras With The Chevalley Property

### 5.1 The Main Theorem

Our main result is the following theorem.

**Theorem 5.1.1** *Let  $H$  be a finite-dimensional triangular Hopf algebra over  $\mathbb{C}$ . Then the following are equivalent:*

1.  *$H$  is a twist of a finite-dimensional triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$  (i.e. of a modified supergroup algebra).*
2.  *$H$  has the Chevalley property.*

The proof of this theorem is contained in the next two subsections.

## 5.2 Local Finite-Dimensional Hopf Superalgebras Are Exterior Algebras

**Theorem 5.2.1** *Let  $\mathcal{H}$  be a local finite-dimensional Hopf superalgebra (not necessarily supercommutative). Then  $\mathcal{H} = \Lambda V^*$  for a finite-dimensional vector space  $V$ . In other words,  $\mathcal{H}$  is the function algebra of an odd vector space  $V$ .*

**Remark 5.2.2** Note that in the commutative case Theorem 5.2.1 is a special case of Proposition 3.2 of [Ko].

**Proof:** It is sufficient to show that  $\mathcal{H}^* = \Lambda V$  for some vector space  $V$ , as  $(\Lambda V)^* = \Lambda V^*$  as Hopf superalgebras. For this, it is sufficient to show that  $\mathcal{H}^*$  is generated by primitive elements, since the sub Hopf superalgebra in  $\mathcal{H}^*$  generated by a basis of the space of primitive elements of  $\mathcal{H}^*$  is clearly a *free* anti-commutative algebra on its generators.

Let  $I$  be the kernel of the counit in  $\mathcal{H}$ . Then  $I = \text{Rad}(\mathcal{H})$  since  $\mathcal{H}$  is local. So in particular there exists a positive integer  $N$  such that for any  $x_1, \dots, x_N \in \mathcal{H}$  one has

$$(x_1 - \varepsilon(x_1)1) \cdots (x_N - \varepsilon(x_N)1) = 0.$$

Let  $\delta_k : \mathcal{H}^* \rightarrow (\mathcal{H}^*)^{\otimes k}$  be the map dual to the map  $\mathcal{H}^{\otimes k} \rightarrow \mathcal{H}$  defined by

$$x_1 \otimes \cdots \otimes x_k \mapsto (x_1 - \varepsilon(x_1)1) \cdots (x_k - \varepsilon(x_k)1)$$

(this map was introduced by Drinfeld in [Dr]). We see that we have a filtration of  $\mathcal{H}^* : \mathcal{H}^* = \cup \mathcal{H}_k^*$ , where  $\mathcal{H}_k^*$  is the kernel of  $\delta_k$  (the  $N$ -th term of this filtration is  $\mathcal{H}^*$ ). In other words,  $\mathcal{H}_k^*$  is the orthogonal complement of  $I^k$ .

Let  $V \subseteq \mathcal{H}^*$  be the space of primitive elements, and  $\mathcal{B} := \Lambda V \subseteq \mathcal{H}^*$  the corresponding Hopf supersubalgebra generated by them. We will prove by induction in  $k$  that  $\mathcal{H}_k^*$  is contained in  $\mathcal{B}$ , which will complete the proof.

The base of induction is obvious (as  $\delta_1(x) = x - \varepsilon(x)$ , hence  $\mathcal{H}_1^* = \mathbf{C}$ ). Suppose the statement is known for  $k = n$ , and let  $a \in \mathcal{H}_{n+1}^*$ . Then it is straightforward to verify that  $j := \Delta(a) - a \otimes 1 - 1 \otimes a \in \mathcal{H}_n^* \otimes \mathcal{H}_n^*$ . So by the induction assumption  $j \in \mathcal{B} \otimes \mathcal{B}$ .

Since the second cohomology of the supercoalgebra  $\Lambda V$  is  $S^2 V$ , any cohomology class can be represented by an even cocycle. Thus, by subtracting from  $a$  an element of  $\Lambda V$ , if needed, we may assume that  $a$  is even.

We claim that  $a \in \mathcal{B}$ . Suppose otherwise. Let  $m \geq 2$  be the smallest integer for which there exists a noncommutative polynomial  $Q(x)$  of elements of  $\Lambda V$  and  $x$ , of degree smaller than  $m$  with respect to  $x$ , such that  $a^m = Q(a)$  (i.e.  $Q(x)$  is a sum of expressions  $b_1 x b_2 x \cdots b_{l-1} x b_l$ ,  $b_i \in \mathcal{B}$ ,  $l \leq m$ ); such polynomial exists because of finite dimensionality.

Since  $a$  is not in  $\Lambda V$ , there exists a linear functional  $f$  on the Hopf algebra which vanishes on  $\Lambda V$  but equals to  $1/m$  on  $a$ . Then applying it in the second component of the equation  $\Delta(a^m) = \Delta(Q(a))$ , we find that  $a^{m-1} = P(a)$ , where  $P$  is a noncommutative polynomial of degree  $< m - 1$ . This is a contradiction, hence  $a \in \mathcal{B}$ . We are done. ■

**Remark 5.2.3** In the appendix we give another proof of Theorem 5.2.1 using the Lifting method of [AS2].

Theorem 5.2.1 will be used in the next subsection, but it also allows one to give the following proof of Corollary 2.3.5.

**Proof of Corollary 2.3.5:** Let  $I$  be the ideal in  $\mathcal{H}^*$  generated by all the odd elements. It is easy to see that this is a Hopf ideal. Consider the Hopf algebra  $E := \mathcal{H}^*/I$  (the even part). This is an *ordinary* commutative Hopf algebra, so  $E = F(G)$  for a suitable finite group  $G$ . Moreover, it is clear that every element of  $I$  is nilpotent, so  $I = \text{Rad}(\mathcal{H}^*)$ . Thus, irreducible  $\mathcal{H}^*$ -modules are 1-dimensional, and are parameterized by  $g \in G$ . Let us call them  $L_g$ . Also, we see that  $G = \mathbf{G}(\mathcal{H})$ .

Let  $P_g$  be the projective cover of the irreducible module  $L_g$ . Then  $\mathcal{H}^* = \bigoplus_g P_g$ , where  $P_g$  are indecomposable two-sided ideals (the ideals are two-sided since the algebra is commutative in the super-sense). In particular,  $P_g$  are local algebras, with 1-dimensional semisimple quotient. Also, we have a natural projection of algebras  $\mathcal{H}^* \rightarrow P_g$  for all  $g$ , in particular  $\mathcal{H}^* \rightarrow P_1$ .

Note that  $\mathcal{H}$  acts on  $\mathcal{H}^*$  on the left and right. In particular, so does the group  $G$ .

**Lemma.** *The following hold:*

1.  $g_1 P_g g_2 = P_{g_1 g g_2}$ .
2.  $\Delta(P_g) \subset \bigoplus_{g_1, g_2: g_1 g_2 = g} P_{g_1} \otimes P_{g_2}$ .

**Proof:** Straightforward. ■

**Corollary.** *The ideal  $\mathcal{I} := \bigoplus_{g \neq 1} P_g$  is a Hopf ideal, and thus  $P_1 = \mathcal{H}^*/\mathcal{I}$  is a Hopf superalgebra.*

Thus,  $P_1^* \subset \mathcal{H}$  is a sub Hopf superalgebra with an action of  $G$ , and we have a factorization  $\mathcal{H} = \mathbf{C}[G] \rtimes P_1^*$ . The Hopf superalgebra  $P_1$  is local, so  $P_1^* = \Lambda V$  by Theorem 5.2.1. This concludes the proof. ■

**Remark 5.2.4** Here is the same proof, described in a more intuitive geometric language. Consider  $\tilde{G} := \text{Spec}(\mathcal{H}^*)$ . This is an affine supergroup scheme. Let  $G \subseteq \tilde{G}$  be the even part of  $\tilde{G}$ . Then  $G$  is a finite group scheme, so by a standard theorem it is a finite group. Let  $V$  be the connected component of the identity in  $\tilde{G}$ . Then the function algebra  $\mathcal{O}(V)$  on  $V$  is

a local finite-dimensional Hopf superalgebra. So by Theorem 5.2.1,  $\mathcal{O}(V) = \Lambda V^*$  for some finite-dimensional vector space  $V$ .

Thus, we have a split exact sequence of algebraic supergroups

$$1 \rightarrow V \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

(it is split because  $G$  is a subgroup of  $\tilde{G}$  complementary to  $V$ ). So  $\tilde{G} = G \ltimes V$ , as desired.

### 5.3 Proof of the Main Theorem

We start by giving a super-analogue of Theorem 3.1 in [G].

**Lemma 5.3.1** *Let  $\mathcal{H}$  be a minimal triangular pointed Hopf superalgebra. Then  $\text{Rad}(\mathcal{H})$  is a Hopf ideal, and  $\mathcal{H}/\text{Rad}(\mathcal{H})$  is minimal triangular.*

**Proof:** The proof is a tautological generalization of the proof of Theorem 3.1 in [G] to the super case.

First of all, it is clear that  $\text{Rad}(\mathcal{H})$  is a Hopf ideal, since its orthogonal complement (the coradical of  $\mathcal{H}^*$ ) is a sub Hopf superalgebra (as  $\mathcal{H}^*$  is isomorphic to  $\mathcal{H}^{cop}$  as a coalgebra, and hence is pointed). Thus, it remains to show that the triangular structure on  $\mathcal{H}$  descends to a minimal triangular structure on  $\mathcal{H}/\text{Rad}(\mathcal{H})$ . For this, it suffices to prove that the composition of the Hopf superalgebra maps

$$\text{Corad}(\mathcal{H}^{*cop}) \hookrightarrow \mathcal{H}^{*cop} \rightarrow \mathcal{H} \rightarrow \mathcal{H}/\text{Rad}(\mathcal{H})$$

(where the middle map is given by the  $R$ -matrix) is an isomorphism. But this follows from the fact that for any surjective coalgebra map  $\eta : C_1 \rightarrow C_2$ , the image of the coradical of  $C_1$  contains the coradical of  $C_2$  ([M], Corollary 5.3.5): One needs to apply this statement to the map  $\mathcal{H}^{*cop} \rightarrow \mathcal{H}/\text{Rad}(\mathcal{H})$ . ■

**Lemma 5.3.2** *Let  $\mathcal{H}$  be a minimal triangular pointed Hopf superalgebra, such that the  $R$ -matrix  $\mathcal{R}$  of  $\mathcal{H}$  is unipotent (i.e.  $\mathcal{R} - 1 \otimes 1$  is 0 in  $\mathcal{H}/\text{Rad}(\mathcal{H}) \otimes \mathcal{H}/\text{Rad}(\mathcal{H})$ ). Then  $\mathcal{H} = \Lambda V$  as a Hopf superalgebra, and  $\mathcal{R} = e^r$ , where  $r \in S^2 V$  is a nondegenerate symmetric (in the usual sense) bilinear form on  $V^*$ .*

**Proof:** By Lemma 5.3.1,  $\text{Rad}(\mathcal{H})$  is a Hopf ideal, and  $\mathcal{H}/\text{Rad}(\mathcal{H})$  is minimal triangular. But the  $R$ -matrix of  $\mathcal{H}/\text{Rad}(\mathcal{H})$  must be  $1 \otimes 1$ , so  $\mathcal{H}/\text{Rad}(\mathcal{H})$  is 1-dimensional. Hence  $\mathcal{H}$  is local, so by Theorem 5.2.1,  $\mathcal{H} = \Lambda V$ . If  $\mathcal{R}$  is a triangular structure on  $\mathcal{H}$  then it comes from an isomorphism  $\Lambda V^* \rightarrow \Lambda V$  of Hopf superalgebras, which is induced by a linear isomorphism  $r : V^* \rightarrow V$ . So  $\mathcal{R} = e^r$ , where  $r$  is regarded as an element of  $V \otimes V$ . Since  $\mathcal{R}\mathcal{R}_{21} = 1$ , we have  $r + r^{21} = 0$  (where  $r^{21} = -r^{op}$  is the opposite of  $r$  in the supersense), so  $r \in S^2 V$ . ■

**Remark 5.3.3** The classification of pointed finite-dimensional Hopf algebras with coradical of dimension 2 is known [CD,N]. In the appendix we use the Lifting method [AS1,AS2] to give an alternative proof. Below we shall need the following more precise version of this result in the triangular case.

**Lemma 5.3.4** *Let  $H$  be a minimal triangular pointed Hopf algebra, whose coradical is  $\mathbf{C}[\mathbb{Z}_2] = \text{sp}\{1, u\}$ , where  $u$  is the Drinfeld element of  $H$ . Then  $H = (\Lambda V)^{\mathcal{J}}$  with the triangular structure of Corollary 3.3.2, where  $\mathcal{J} = e^{r/2}$ , with  $r \in S^2V$  a nondegenerate element. In particular,  $H$  is a twist of a modified supergroup algebra.*

**Proof:** Let  $\mathcal{H}$  be the associated triangular Hopf superalgebra to  $H$ , as described in Theorem 3.3.1. Then the  $R$ -matrix of  $\mathcal{H}$  is unipotent, because it turns into  $1 \otimes 1$  after killing the radical.

Let  $\mathcal{H}_m$  be the minimal part of  $\mathcal{H}$ . By Lemma 5.3.2,  $\mathcal{H}_m = \Lambda V$  and  $\mathcal{R} = e^r$ ,  $r \in S^2V$ . So if  $\mathcal{J} := e^{r/2}$  then  $\mathcal{H}^{\mathcal{J}^{-1}}$  has  $R$ -matrix equal to  $1 \otimes 1$ . Thus,  $\mathcal{H}^{\mathcal{J}^{-1}}$  is cocommutative, so by Corollary 2.3.5, it equals  $\mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$ . Hence  $\mathcal{H} = \mathbf{C}[\mathbb{Z}_2] \ltimes (\Lambda V)^{\mathcal{J}}$ , and the result follows from Proposition 3.2.1. ■

We shall need the following lemma.

**Lemma 5.3.5** *Let  $B \subseteq A$  be finite-dimensional associative unital algebras. Then any simple  $B$ -module is a constituent (in the Jordan-Holder series) of some simple  $A$ -module.*

**Proof:** Since  $A$ , considered as a  $B$ -module, contains  $B$  as a  $B$ -module, any simple  $B$ -module is a constituent of  $A$ .

Decompose  $A$  (in the Grothendieck group of  $A$ ) into simple  $A$ -modules:  $A = \sum V_i$ . Further decomposing as  $B$ -modules, we get  $V_i = \sum W_{ij}$ , and hence  $A = \sum_i \sum_j W_{ij}$ . Now, by Jordan-Holder theorem, since  $A$  (as a  $B$ -module) contains all simple  $B$ -modules, any simple  $B$ -module  $X$  is in  $\{W_{ij}\}$ . Thus,  $X$  is a constituent of some  $V_i$ , as desired. ■

**Proposition 5.3.6** *Any minimal triangular Hopf algebra  $H$  with the Chevalley property is a twist of a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ .*

**Proof:** By Proposition 4.2, the coradical  $H_0$  of  $H$  is a Hopf subalgebra, since  $H \simeq H^{*cop}$ , being minimal triangular. Consider the Hopf algebra map  $\varphi : H_0 \rightarrow H^{*cop}/\text{Rad}(H^{*cop})$ , given by the composition of the following maps:

$$H_0 \hookrightarrow H \simeq H^{*cop} \rightarrow H^{*cop}/\text{Rad}(H^{*cop}),$$

where the second map is given by the  $R$ -matrix. We claim that  $\phi$  is an isomorphism. Indeed,  $H_0$  and  $H^{*cop}/\text{Rad}(H^{*cop})$  have the same dimension, since  $\text{Rad}(H^{*cop}) = (H_0)^\perp$ , and  $\phi$  is injective, since  $H_0$  is semisimple by [LR]. Let  $\pi : H \rightarrow H_0$  be the associated projection.

We see, arguing exactly as in [G, Theorem 3.1], that  $H_0$  is also minimal triangular, say with  $R$ -matrix  $R_0$ .

Now, by [EG1, Theorem 2.1], we can find a twist  $J$  in  $H_0 \otimes H_0$  such that  $(H_0)^J$  is isomorphic to a group algebra and has  $R$ -matrix  $(R_0)^J$  of rank  $\leq 2$ . Notice that here we are relying on Deligne's theorem, as mentioned in the introduction.

Let us now consider  $J$  as an element of  $H_0 \otimes H_0$  and the twisted Hopf algebra  $H^J$ , which is again triangular.

The projection  $\pi : H^J \rightarrow (H_0)^J$  is still a Hopf algebra map, and sends  $R^J$  to  $(R_0)^J$ . It induces a projection  $(H^J)_m \rightarrow \mathbf{C}[\mathbb{Z}_2]$ , whose kernel  $K_m$  is contained in the kernel of  $\pi$ . Because any simple  $(H^J)_m$ -module is contained as a constituent in a simple  $H$ -module (see Lemma 5.3.5),  $K_m = \text{Rad}((H^J)_m)$ . Hence,  $(H^J)_m$  is minimal triangular and  $(H^J)_m/\text{Rad}((H^J)_m) = (\mathbf{C}[\mathbb{Z}_2], R_u)$ . It follows, again by minimality, that  $(H^J)_m$  is also pointed with coradical isomorphic to  $\mathbf{C}[\mathbb{Z}_2]$ . So by Lemma 5.3.4,  $(H^J)_m$ , and hence  $H^J$ , can be further twisted into a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ , as desired. ■

Now we can prove the main theorem.

**Proof of Theorem 5.1.1:** (2.  $\Rightarrow$  1.) By Proposition 4.2,  $H/\text{Rad}(H)$  is a semisimple Hopf algebra. Let  $H_m$  be the minimal part of  $H$ , and  $H'_m$  be the image of  $H_m$  in  $H/\text{Rad}(H)$ . Then  $H'_m$  is a semisimple Hopf algebra.

Consider the kernel  $K$  of the projection  $H_m \rightarrow H'_m$ . Then  $K = \text{Rad}(H) \cap H_m$ . This means that any element  $k \in K$  is zero in any simple  $H$ -module. This implies that  $k$  acts by zero in any simple  $H_m$ -module, since by Lemma 5.3.5, any simple  $H_m$ -module occurs as a constituent of some simple  $H$ -module. Thus,  $K$  is contained in  $\text{Rad}(H_m)$ . On the other hand,  $H_m/K$  is semisimple, so  $K = \text{Rad}(H_m)$ . This shows that  $\text{Rad}(H_m)$  is a Hopf ideal. Thus,  $H_m$  is minimal triangular satisfying the conditions of Proposition 5.3.6. By Proposition 5.3.6,  $H_m$  is a twist of a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ . Hence  $H$  is a twist of a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$  (by the same twist), as desired.

(1.  $\Rightarrow$  2.) By assumption,  $\text{Rep}(H)$  is equivalent to  $\text{Rep}(\tilde{G})$  for some supergroup  $\tilde{G}$  (as a tensor category without braiding). But we know that supergroup algebras have the Chevalley property, since, modulo their radicals, they are group algebras. This concludes the proof of the main theorem. ■

**Remark 5.3.7** Notice that it follows from the proof of the main theorem that any triangular Hopf algebra with the Chevalley property can be obtained by twisting of a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$  by an *even* twist.

**Definition 5.3.8** *If a triangular Hopf algebra  $H$  over  $\mathbf{C}$  satisfies condition 1. or 2. of Theorem 5.1.1, we will say that  $H$  is of supergroup type.*

## 5.4 Corollaries of the Main Theorem

**Corollary 5.4.1** *A finite-dimensional triangular Hopf algebra  $H$  is of supergroup type if and only if so is its minimal part  $H_m$ .*

**Proof:** If  $H$  is of supergroup type then  $\text{Rad}(H)$  is a Hopf ideal, so like in the proof of Theorem 5.1.1 (2.  $\Rightarrow$  1.) we conclude that  $\text{Rad}(H_m)$  is a Hopf ideal, i.e.  $H_m$  is of supergroup type.

Conversely, if  $H_m$  is of supergroup type then  $H_m$  is a twist of a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ . Hence  $H$  is a twist of a triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$  (by the same twist), so  $H$  is of supergroup type. ■

**Corollary 5.4.2** *A finite-dimensional triangular Hopf algebra whose coradical is a Hopf subalgebra is of supergroup type. In particular, this is the case for finite-dimensional triangular pointed Hopf algebras.*

**Proof:** This follows from Corollary 5.4.1. ■

**Corollary 5.4.3** *Any finite-dimensional triangular basic Hopf algebra is of supergroup type.*

**Proof:** A basic Hopf algebra automatically has the Chevalley property since all its irreducible modules are 1-dimensional. Hence the result follows from the main theorem. ■

## 5.5 Questions

The above results motivate the following question.

**Question 5.5.1** *Does any finite-dimensional triangular Hopf algebra over  $\mathbf{C}$  have the Chevalley property (i.e. is of supergroup type)? Is it true under the assumption that  $S^4 = Id$  or at least under the assumption that  $u^2 = 1$ ?*

**Remark 5.5.2** Recall [G] that it is not known whether any finite-dimensional triangular Hopf algebra over  $\mathbf{C}$  has the property  $u^2 = 1$  or at least  $S^4 = Id$ . It is also not known if  $S^4 = Id$  implies  $u^2 = 1$  for triangular Hopf algebras. However, it is clear that for finite-dimensional triangular Hopf algebras  $H$  of supergroup type,  $u^2 = 1$  (and hence  $S^4 = Id$ ).

Indeed, since  $S^2 = Id$  on the semisimple part of  $H$ ,  $u$  acts by a scalar in any irreducible representation of  $H$ . In fact, since  $\text{tr}(u) = \text{tr}(u^{-1})$ , we have that  $u = 1$  or  $u = -1$  on any irreducible representation of  $H$ , and hence  $u^2 = 1$  on any irreducible representation of  $H$ . Thus,  $u^2$  is unipotent. But it is of finite order (as it is a grouplike element), so it is equal to 1 as desired.

**Remark 5.5.3** Note that the answer to question 5.5.1 is negative in the infinite-dimensional case. Namely, although the answer is positive in the cocommutative case (by [C]), it is negative already for triangular Hopf algebras with  $R$ -matrix of rank 2, which correspond to cocommutative Hopf superalgebras. Indeed, let us take the cocommutative Hopf superalgebra  $\mathcal{H} := U(\mathfrak{gl}(n|n))$  (for the definition of the Lie superalgebra  $\mathfrak{gl}(n|n)$ , see [Ka, p.29]). The associated triangular Hopf algebra  $\overline{\mathcal{H}}$  does not have the Chevalley property, since it is well known that Chevalley theorem fails for Lie superalgebras (e.g.  $\mathfrak{gl}(n|n)$ ); more precisely, already the product of the vector and covector representations for this Lie superalgebra is not semisimple.

**Remark 5.5.4** It follows from Corollary 5.4.1 that a positive answer to Question 5.5.1 in the minimal case would imply the general positive answer.

Here is a generalization of Question 5.5.1.

**Question 5.5.5** Does any  $\mathbf{C}$ -linear abelian symmetric rigid tensor category, with  $\text{End}(\mathbf{1}) = \mathbf{C}$  and finitely many simple objects, have the Chevalley property?

Even a more ambitious question:

**Question 5.5.6** Is such a category equivalent to the category of representations of a finite-dimensional triangular Hopf algebra with  $R$ -matrix of rank  $\leq 2$ ? In particular, is it equivalent to the category of representations of a supergroup, as a category without braiding? Are these statements valid at least for categories with Chevalley property? For semisimple categories?

## 6 Finite-Dimensional Cotriangular Pointed Hopf Algebras are Generated by Grouplikes and Skewprimitives

There is a conjecture (see [AS2]) that any finite-dimensional pointed Hopf algebra over  $\mathbf{C}$  is generated by grouplike and skew primitive elements. Here we confirm it in the cotriangular case.

**Theorem 6.1** *A finite-dimensional cotriangular pointed Hopf algebra  $H$  over  $\mathbf{C}$  is generated by grouplike and skew primitive elements.*

In order to prove the theorem, we will need the following lemma.

**Lemma 6.2** *Let  $H$  be a finite-dimensional pointed Hopf algebra or superalgebra. Then the following are equivalent:*

1.  *$H$  is generated by grouplike and skew primitive elements.*
2. *There exists a faithful  $H^*$ -module which is a direct sum of tensor products of  $H^*$ -modules of dimension 2.*

**Proof:** Irreducible  $H^*$ -modules are 1-dimensional, so a 2-dimensional representation has the form

$$a \mapsto \begin{pmatrix} p(a) & r(a) \\ 0 & q(a) \end{pmatrix}, \quad a \in H^*,$$

where  $p, q$  are characters (i.e. belong to  $\mathbf{G}(H)$ ), and  $r$  is a  $q : p$  skew primitive. Conversely, such a 2-dimensional representation exists for any skew primitive element. Matrix elements of tensor products of representations of  $H^*$  are products of the matrix elements of these representations (as elements of  $H$ ). This implies the lemma. ■

Now we are ready to give a proof of Theorem 6.1.

We need to show that  $H^*$  has a faithful representation which is a direct sum of products of 2-dimensional representations. By [G], the Drinfeld element  $u$  of  $H^*$  satisfies  $u^2 = 1$ . Hence by Theorem 3.1.1, we can replace  $H^*$  by the corresponding Hopf superalgebra  $\widetilde{H}^*$  (as this does not change the representation category as a tensor category). But  $H^*$  is basic triangular, which means, by Corollary 5.4.3, that  $\widetilde{H}^*$  is twist equivalent to a supergroup algebra  $B$ . Thus by Lemma 6.2, it suffices to show that  $B^*$  (the dual of  $B$ ) is generated by grouplikes and skew primitives.

But  $B = \mathbf{C}[G] \ltimes \Lambda V$  where  $G$  is abelian. Thus,  $V$  is decomposed in the direct sum of eigenspaces for  $G$ . So let  $v_1, \dots, v_n$  be a basis of  $V$ , such that  $gv_i g^{-1} = \chi_i(g)v_i$ , where  $\chi_i$  are some characters of  $G$ . Using this presentation of  $B$ , it is easy to compute its dual  $B^*$  and show that it is generated as an algebra by  $G^\vee$  (the character group) and  $\chi_i : 1$  skew primitive elements  $\xi_i$ ,  $i = 1, \dots, n$ . We are done. ■

**Corollary 6.3** *Theorem 5.1 of [G] gives the classification of all minimal triangular pointed Hopf algebras.*

**Proof:** Since minimal triangular pointed Hopf algebras are also cotriangular, by Theorem 6.1, they are generated by grouplikes and skewprimitives (which answers a question from [G]). On the other hand, [G, Theorem 5.1] gives a classification of minimal triangular Hopf algebras which are generated by grouplikes and skewprimitives. ■

**Remark 6.4** Lemma 6.2 implies that if  $H_1, H_2$  are finite-dimensional pointed Hopf algebras, and the comultiplication of  $H_1^*$  is obtained by conjugating that of  $H_2^*$  by an invertible element (not necessarily a twist), then  $H_1$  is generated by grouplike and skew primitive elements if and only if so is  $H_2$ .

## 7 Categorical Dimensions in Symmetric Categories With Finitely Many Irreducibles are Integers

In this paper we classified finite-dimensional triangular Hopf algebras with the Chevalley property. In conclusion, let us give one result that is valid in a greater generality: for any finite-dimensional triangular Hopf algebra, and even for any symmetric rigid category with finitely many irreducible objects.

Let  $\mathcal{C}$  be a  $\mathbf{C}$ -linear abelian symmetric rigid category with  $\mathbf{1}$  as its unit object, and suppose that  $\text{End}(\mathbf{1}) = \mathbf{C}$ . Recall that there is a natural notion of dimension in  $\mathcal{C}$ , generalizing the ordinary dimension of an object in  $\text{Vect}$ . Let  $\beta$  denote the commutativity constraint in  $\mathcal{C}$ , and for an object  $V$ , let  $ev_V, coev_V$  denote the associated evaluation and coevaluation morphisms.

**Definition 7.1** [DM] *The categorical dimension  $\dim_c(V) \in \mathbf{C}$  of  $V \in \text{Ob}(\mathcal{C})$  is the morphism*

$$\dim_c(V) : \mathbf{1} \xrightarrow{ev_V} V \otimes V^* \xrightarrow{\beta_{V,V^*}} V^* \otimes V \xrightarrow{coev_V} \mathbf{1}. \quad (10)$$

The main result of this section is the following:

**Theorem 7.2** *In any  $\mathbf{C}$ -linear abelian symmetric rigid tensor category  $\mathcal{C}$  with finitely many irreducible objects, the categorical dimensions of objects are integers.*

**Proof:** First note that the categorical dimension of any object  $V$  of  $\mathcal{C}$  is an algebraic integer. Indeed, let  $V_1 \dots, V_n$  be the irreducible objects of  $\mathcal{C}$ . Then  $\{V_1 \dots, V_n\}$  is a basis of the Grothendieck ring of  $\mathcal{C}$ . Write  $V \otimes V_i = \sum_j N_{ij}(V) V_j$  in the Grothendieck ring. Then  $N_{ij}(V)$  is a matrix with integer entries, and  $\dim_c(V)$  is an eigenvalue of this matrix. Thus,  $\dim_c(V)$  is an algebraic integer.

Now, if  $\dim_c(V) = d$  then it is easy to show (see e.g. [De2]) that

$$\dim_c(S^k V) = d(d+1) \cdots (d+k-1)/k!,$$

and

$$\dim_c(\Lambda^k V) = d(d-1) \cdots (d-k+1)/k!,$$

hence they are also algebraic integers. So the theorem follows from:

**Lemma.** *Suppose  $d$  is an algebraic integer such that  $d(d+1)\cdots(d+k-1)/k!$  and  $d(d-1)\cdots(d-k+1)/k!$  are algebraic integers for all  $k$ . Then  $d$  is an integer.*

**Proof:** Let  $Q$  be the minimal monic polynomial of  $d$  over  $\mathbb{Z}$ . Since  $d(d-1)\cdots(d-k+1)/k!$  is an algebraic integer, so are the numbers  $d'(d'-1)\cdots(d'-k+1)/k!$ , where  $d'$  is any algebraic conjugate of  $d$ . Taking the product over all conjugates, we get that

$$N(d)N(d-1)\cdots N(d-k+1)/(k!)^n$$

is an integer, where  $n$  is the degree of  $Q$ . But  $N(d-x) = (-1)^n Q(x)$ . So we get that  $Q(0)Q(1)\cdots Q(k-1)/(k!)^n$  is an integer. Similarly from the identity for  $S^k V$ , it follows that  $Q(0)Q(-1)\cdots Q(1-k)/(k!)^n$  is an integer. Now, without loss of generality, we can assume that  $Q(x) = x^n + ax^{n-1} + \dots$ , where  $a \leq 0$  (otherwise replace  $Q(x)$  by  $Q(-x)$ ; we can do it since our condition is symmetric under this change). Then for large  $k$ , we have  $Q(k-1) < k^n$ , so the sequence  $b_k := Q(0)Q(1)\cdots Q(k-1)/(k!)^n$  is decreasing in absolute value or zero starting from some place. But a sequence of integers cannot be strictly decreasing in absolute value forever. So  $b_k = 0$  for some  $k$ , hence  $Q$  has an integer root. This means that  $d$  is an integer (i.e.  $Q$  is linear), since  $Q$  must be irreducible over the rationals. This concludes the proof of the lemma, and hence of the theorem. ■

**Corollary 7.3** *For any triangular Hopf algebra  $H$  (not necessarily finite-dimensional), the categorical dimensions of its finite-dimensional representations are integers.*

**Proof:** It is enough to consider the minimal part  $H_m$  of  $H$  which is finite-dimensional, since  $\dim_c(V) = \text{tr}(u|_V)$  for any module  $V$  (where  $u$  is the Drinfeld element of  $H$ ), and  $u \in H_m$ . Hence the result follows from Theorem 7.2. ■

**Remark 7.4** Theorem 7.2 is false without the finiteness conditions. In fact, in this case any complex number can be a dimension, as is demonstrated in examples constructed by Deligne [De1, p.324-325]. Also, it is well known that the theorem is false for ribbon, nonsymmetric categories (e.g. for fusion categories of semisimple representations of finite-dimensional quantum groups at roots of unity, where dimensions can be irrational algebraic integers).

**Remark 7.5** Note that Theorem 7.2 can be regarded as a piece of supporting evidence for a positive answer to Question 5.5.6.

**Remark 7.6** In any rigid braided tensor category with finitely many irreducible objects, one can define the Frobenius-Perron dimension of an object  $V$ ,  $\text{FPdim}(V)$ , to be the largest positive eigenvalue of the matrix of multiplication by  $V$  in the Grothendieck ring. This

dimension is well defined by the Frobenius-Perron theorem, and has the usual additivity and multiplicativity properties. For example, for the category of representations of a quasi-Hopf algebra, it is just the usual dimension of the underlying vector space. If the answer to Question 5.5.6 is positive then  $\text{FPdim}(V)$  for symmetric categories is always an integer, which is equal to  $\dim_c(V)$  modulo 2. It would be interesting to check this, at least in the case of modules over a triangular Hopf algebras, when the integrality of  $\text{FPdim}$  is automatic (so only the mod 2 congruence has to be checked).

## 8 Appendix: On Pointed Hopf Algebras

In this appendix we use the Lifting method [AS1, AS2] to give other proofs of Theorem 5.2.1 and Corollary 6.3, and a generalization of Lemma 5.3.4.

By a *braided Hopf algebra* we shall mean a Hopf algebra in the braided tensor category of Yetter-Drinfeld modules over a group algebra  $\mathbf{C}[\Gamma]$ , where  $\Gamma$  is a finite abelian group. For example, one can endow the exterior algebra  $\Lambda V$  with the structure of a braided Hopf algebra, as follows. Let  $x_1, \dots, x_N$  be a basis of  $V$  and let there be given  $g_1, \dots, g_N \in \Gamma$  and  $\chi_1, \dots, \chi_N \in \Gamma^\vee$  such that

$$\chi_i(g_j) = -1, \quad 1 \leq i, j \leq N.$$

Then  $V$  is a Yetter-Drinfeld module over  $\mathbf{C}[\Gamma]$  where the action and coaction of  $\Gamma$  on  $x_i$  are given by  $\chi_i$  and  $g_i$  respectively. These action and coaction extend to  $\Lambda V$ , and turn  $\Lambda V$  into a braided Hopf algebra.

**Lemma 8.1** *Let  $R = \bigoplus_{n \geq 0} R(n)$  be a graded braided Hopf algebra such that  $R(0) = \mathbf{C}$ ,  $R(1) \simeq V$  as a Yetter-Drinfeld module (with the assumptions above), and  $R$  is generated by  $R(1)$ . Then  $R$  is isomorphic to  $\Lambda V$  as a graded braided Hopf algebra.*

**Proof:** It is known, and easy to see, that  $\Lambda V$  satisfies all the hypotheses that  $R$  does, plus that the primitive elements are concentrated in degree one:  $\mathbf{P}(\Lambda V) = \Lambda V(1) = V$  (see for instance [AS1, Section 3]). In other words,  $\Lambda V$  is the Nichols algebra of  $V$ , and there exists a surjective homomorphism of graded braided Hopf algebras  $R \rightarrow \Lambda V$  which is the identity in degree one (see for instance [AS2, Lemma 5.5]). On the other hand, it is clear that  $\Lambda V$  can be presented by generators  $x_1, \dots, x_N$  with relations

$$x_i x_j + x_j x_i = 0, \quad 1 \leq i, j \leq N. \tag{11}$$

So in particular  $x_i^2 = 0$  for all  $i$ . It is thus enough to show that equations (11) also hold in  $R$ , with an evident abuse of notation. But  $x_i x_j + x_j x_i$  is a primitive element of  $R$ , whose action is given by the character  $\chi_i \chi_j$ , and whose coaction is given by  $g_i g_j$ . Since  $\chi_i \chi_j(g_i g_j) = 1$  and  $R$  is finite-dimensional,  $x_i x_j + x_j x_i = 0$  in  $R$  by [AS1, Lemma 3.1]. ■

Let  $H$  be a finite-dimensional pointed Hopf algebra such that  $\mathbf{G}(H)$  is isomorphic to  $\Gamma$ . We recall that the Lifting method [AS1, AS2] attaches to  $H$  several invariants:

- The graded Hopf algebra  $\text{gr}H$ , associated to the coradical filtration of  $H$ .
- A graded braided Hopf algebra  $R$ ; the coinvariants of the homogeneous projection from  $\text{gr}H$  to  $\mathbf{C}[\Gamma]$ .
- A Yetter-Drinfeld module  $W := R(1)$  over  $\mathbf{C}[\Gamma]$ , called the infinitesimal braided vector space of  $H$ .

We conclude immediately from Lemma 8.1:

**Corollary 8.2** *Let  $H$  be a finite-dimensional pointed Hopf algebra such that  $\mathbf{G}(H)$  is isomorphic to  $\Gamma$ . Assume that the infinitesimal braiding of  $H$  is isomorphic to  $V$  as above. Then  $H$  is generated by grouplike and skewprimitive elements. ■*

**Remark 8.3** Notice that Corollary 8.2 allows one to give an alternative proof of Corollary 6.3. For, Lemmas 5.3 and 5.4 in [G] imply that the infinitesimal braiding of any minimal triangular pointed Hopf algebra is isomorphic to a  $V$  as above.

Assume now that  $\Gamma = \mathbb{Z}_2$ . There is only one possible choice for  $V$  as above, namely  $g_1 = \dots = g_N = u$  and  $\chi_1 = \dots = \chi_N = \text{the sign}$ . This gives the Hopf superalgebra as explained in Section 5. Let now  $H$  be a finite-dimensional pointed Hopf algebra such that  $\mathbf{G}(H)$  is isomorphic to  $\mathbb{Z}_2$ . Then the infinitesimal braiding of  $H$  is isomorphic to  $V$  as above by [AS1, Lemma 3.1] again, for some natural number  $N$ . The Lifting method gives a very direct proof of the following well-known result.

**Theorem 8.4** [N, Th. 4.2.1], [CD] *If  $H$  is a finite-dimensional pointed Hopf algebra with  $\mathbf{G}(H)$  isomorphic to  $\mathbb{Z}_2$ , then  $H \simeq \mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$ .*

**Proof:** By the above remarks and Corollary 8.2, we know that  $\text{gr}H \simeq \mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$  for some  $V$ . The fact that  $H \simeq \text{gr}H$  ("there are no liftings" in the jargon of the Lifting method) is a particular case of the main result [AS1, Th. 5.5]. ■

We can now give another proof of Theorem 5.2.1.

It is enough to show that  $\mathcal{H}^* = \Lambda V$  for some  $V$  as above, since  $(\Lambda V)^* = \Lambda V^*$  as Hopf superalgebras. By the hypothesis, the coradical of  $\mathcal{H}^*$  is trivial:  $\text{Corad}(\mathcal{H}^*) = \mathbf{C}1$ . We can consider the biproduct  $H := \mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}^*$ ; that is,  $H = \overline{\mathcal{H}}$  in our notation. We claim that  $H$  is a finite-dimensional pointed Hopf algebra with  $\mathbf{G}(H)$  isomorphic to  $\mathbb{Z}_2$ . Indeed, the filtration

$$\mathbf{C}[\mathbb{Z}_2] \subset \mathbf{C}[\mathbb{Z}_2] \ltimes (\mathcal{H}^*)_1 \subset \dots \subset \mathbf{C}[\mathbb{Z}_2] \ltimes (\mathcal{H}^*)_j \subset \dots$$

is a coalgebra filtration, where  $1 \subset (\mathcal{H}^*)_1 \subset \dots \subset (\mathcal{H}^*)_j \subset \dots$  is the coradical filtration of  $\mathcal{H}^*$ . Hence  $\mathbf{C}[\mathbb{Z}_2]$  contains the coradical of  $H$  and the other inclusion is evident.

It follows then from Theorem 8.4 that  $H \simeq \mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$ . By [AS2, Lemma 6.2],  $\mathcal{H}^* \simeq \Lambda V$  as braided Hopf algebras, that is as Hopf superalgebras. ■

## References

- [AS1] N. Andruskiewitsch and H.-J. Schneider, Lifting of Quantum Linear Spaces and Pointed Hopf Algebras of order  $p^3$ , *J. Algebra* **209** (1998), 658-691.
- [AS2] N. Andruskiewitsch and H.-J. Schneider, Finite quantum groups and Cartan matrices, *Adv. in Math.*, to appear.
- [C] C. Chevalley, Theory of Lie groups, v.III, 1951 (in French).
- [CD] S. Caenepeel and S. Dăscălescu, On pointed Hopf algebras of dimension  $2^n$ , *Bull. London Math. Soc.* **31** (1999), 17-24
- [De1] P. Deligne, La série exceptionnelle de groupes de Lie. (French) [The exceptional series of Lie groups], *C. R. Acad. Sci. Paris Ser. I Math.* **322** (1996), no.4, 321–326.
- [De2] P. Deligne, Categories Tannakiennes, In The Grothendick Festschrift, Vol. II, *Prog. Math.* **87** (1990), 111-195.
- [DM] P. Deligne and J. Milne, Tannakian Categories, Lecture Notes in Mathematics **900**, 101-228, 1982.
- [Dr] V. Drinfeld, Quantum Groups, *Proceedings of the International Congress of Mathematicians, Berkeley* (1987), 798-820.
- [EG1] P. Etingof and S. Gelaki, Some Properties of Finite Dimensional Semisimple Hopf Algebras, *Mathematical Research Letters* **5** (1998), 191-197.
- [EG2] P. Etingof and S. Gelaki, The Classification of Triangular Semisimple and Cosemisimple Hopf Algebras Over an Algebraically Closed Field, *International Mathematics Research Notices* **5** (2000), 223-234.
- [G] S. Gelaki, Some Properties and Examples of Pointed Triangular Hopf Algebras, *Mathematical Research Letters* **6** (1999), 563-572; see corrected version at [math.QA/9907106](http://math.QA/9907106).
- [Ka] V. Kac, Lie superalgebras, *Advances in Math.* **26**, No.1, 1977.
- [Ko] B. Kostant, Graded manifolds, graded Lie theory, and prequantization, *Differ. geom. Meth. math. Phys.*, Proc. Symp. Bonn 1975, *Lect. Notes Math.* **570** (1977), 177-306.
- [L] G. Lusztig, Finite Dimensional Hopf Algebras Arising from Quantized Universal Enveloping Algebras, *J. of the A.M.S* Vol.3, No.1 (1990), 257-296.

- [LR] R. Larson and D. Radford, Semisimple Cosemisimple Hopf Algebras, *American J. of Mathematics* **110** (1988), 187-195.
- [M] S. Montgomery, Hopf algebras and their actions on rings, AMS, 1994.
- [N] W.D. Nichols, Bialgebras of type one, *Comm. Alg.* **6** (1978), 1521-1552.
- [R1] D.E. Radford, Minimal quasitriangular Hopf algebras, *Journal of Algebra* **157** (1993), 285-315.
- [R2] D.E. Radford, The Structure of Hopf Algebras with a Projection, *Journal of Algebra* **2** (1985), 322-347.
- [S] M. Sweedler, Hopf Algebras, Benjamin Press, 1968.